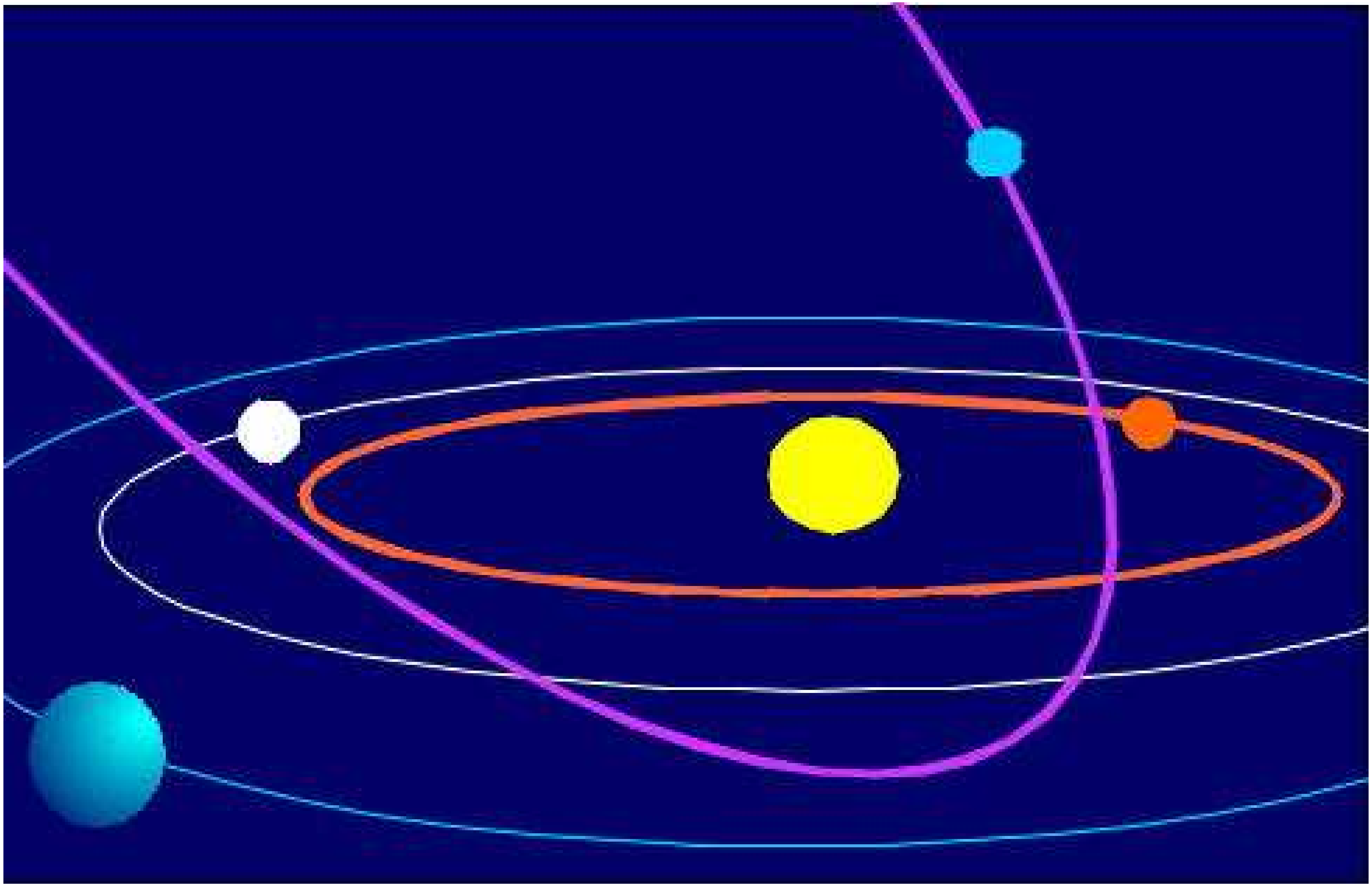
# Celestial mechanics



Contribution to computational physics 6.5.2008
by Dr. Christian Anders

# Owerview for Celestial mechanics

1. Kepler's and Newtons's laws,
   equation of motion

2. Numerical solution: Runge-Kutta-Algorithmus

3. Orbit types

4. Restricted three-body-problem

5. Interactive simulation of 3 body interaction

   Trojans in solar-system,

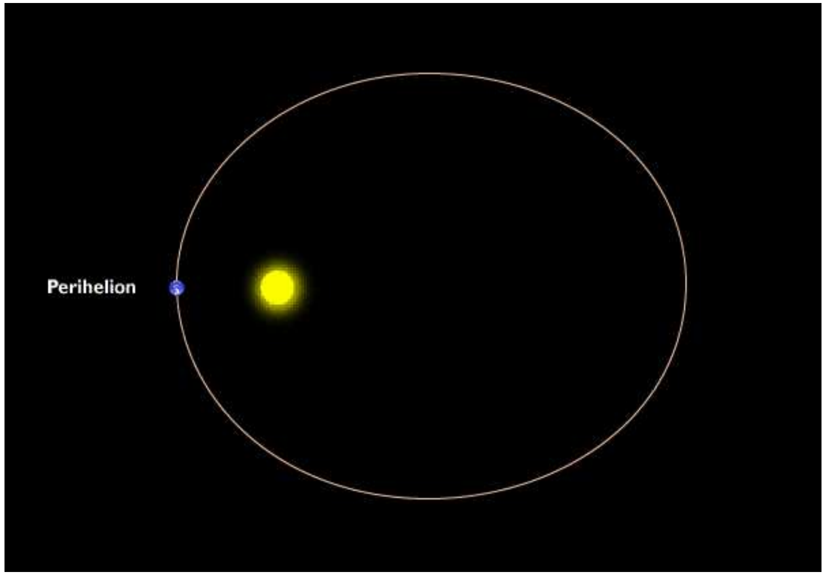
   Comet capture, Voyager

   Double star systems

6. Literature

# 1. Theory of Celestial mechanics

**Keplers laws**




infered from observations
of Tycho Brahe

I+II [Astronomia nova, **1609**]

**I. The planets move in ellipses with the sun in one of the foci.**

**II. The pointer to a planet moves over equal areas in equal times**
(<=> Conservation of angular Momentum)

**III.** [Harmonices mundi, **1619**]

$a^3/T^2 = const$ (approximatively)

$$\frac{a_1^3}{a_2^3} = \frac{T_1^2(M_S + m_1)}{T_2^2(M_S + m_2)}$$ (exactly)

# Newtonian mechanics

- **Law of Gravitation** [Newton, 1687]**:**
  $F = -GMm/r^2$ , $F = ma$
- accelleration of object $i$: $\vec{a}_i = \frac{d^2}{dt^2}\vec{x}_i = -G\sum_{i \neq j} \frac{m_j(\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3}$
- system of $n$ 2nd order diff. equations ! ($xyz$->3$n$)
- phase-space: $x$ and $v$ as independent variables
  composed vector
  $$\vec{y} \equiv (\vec{x}, \vec{v}), \vec{x} = (\vec{x}_1, \ldots, \vec{x}_n), \vec{v} = (\vec{v}_1, \ldots, \vec{v}_n)$$
- 2$n$ 1st order diff. equation to integrate ($xyz$->6$n$)
  $$\frac{d\vec{y}}{dt} = \begin{pmatrix} \frac{d\vec{x}}{dt} \\ \frac{d\vec{v}}{dt} \end{pmatrix} = \begin{pmatrix} \vec{v} \\ \vec{a}(\vec{x}) \end{pmatrix} = \vec{g}(\vec{y})$$
  for composed derrivative $g$ (*)

# 2. Numerical Solution

- Simple Euler recursion for time step $h$:

$$\vec{y}(t+h) = \vec{y}(t) + h\vec{g}(\vec{y}(t),t) + o(h^2)$$

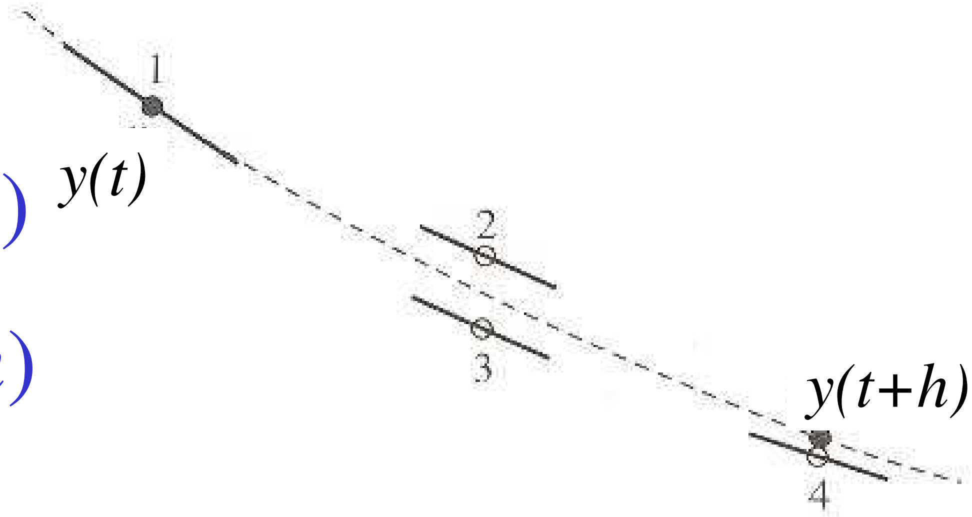
- **Runge-Kutta-Algorithm**
[Press et al., Numerical Recipes in C etc.]

$$\vec{k}_1 = h \cdot \vec{g}(\vec{y}(t),t)$$

$$\vec{k}_2 = h \cdot \vec{g}(\vec{y}(t) + \tfrac{1}{2}\vec{k}_1, t + \tfrac{1}{2}h)$$

$$\vec{k}_3 = h \cdot \vec{g}(\vec{y}(t) + \tfrac{1}{2}\vec{k}_2, t + \tfrac{1}{2}h)$$

$$\vec{k}_4 = h \cdot \vec{g}(\vec{y}(t) + \vec{k}_3, t + h)$$



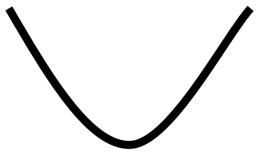

$$\vec{y}(t+h) = \vec{y}(t) + \tfrac{1}{6}\left[\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right] + o(h^4)$$

accurancy $o(h^4)$ by intermediate derrivatives $g \to k$
Further improvement: adaptive Stepzize

# 3. Orbit types

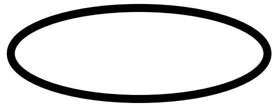
- Energy of planet in two body system
- $E = \frac{1}{2}\mu v^2 - \frac{GMm}{r}, \mu = \frac{Mm}{M+m}$ $m$: Planet, $M$: Star
- Semimayoraxis $a = -\frac{1}{2}GMm/E$
- Type of orbit

$E > 0$ => hyperbolical

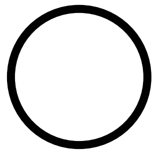


$E = 0$ => parabolical

$E < 0$ => elliptical



$E = -\frac{1}{2}\frac{GMm}{r}$ => circular orbit if $v \perp r$



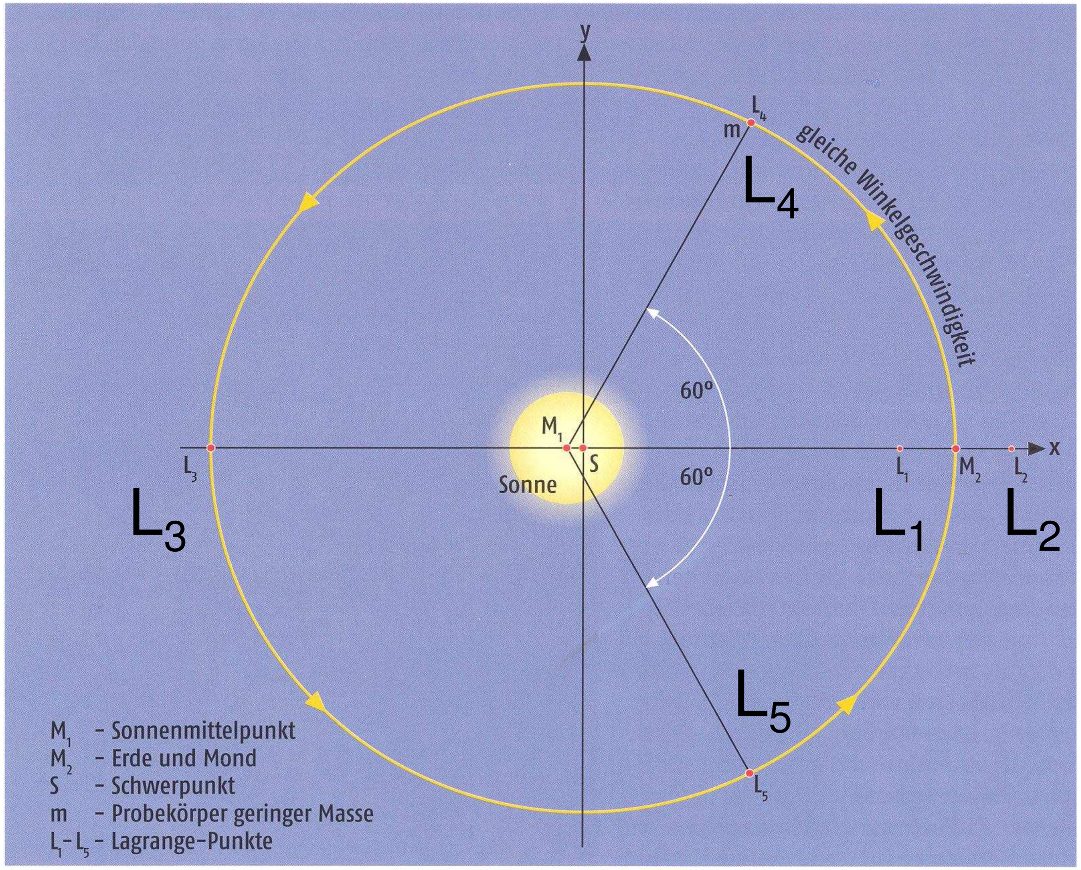
$$\frac{\mu v^2}{r} = \frac{GMm}{r^2} \Rightarrow v = \sqrt{\frac{GM}{r}} \quad (m \ll M)$$

- Correction for $m \rightarrow M$ in COM System

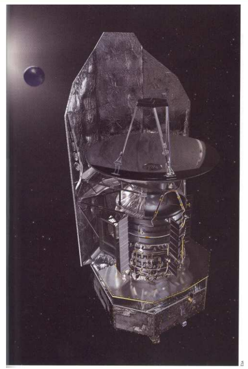
$$v_p = \sqrt{\frac{GM}{r}\frac{M}{m+M}}, v_S = \sqrt{\frac{Gm}{r}\frac{m}{m+M}}$$

# 4. Restricted Three-body-problem

- [Lagrange 1772] analytically solvable: heavy $M_1$ and $M_2$ with $M_2 < 38{,}5$‰ $(M_1+M_2)$ and a light $m \ll (M_1+M_2)$
- **Observing in a frame rotating with the baseline $M_1$-$M_2$** $M_1$, $M_2$ and $m$ remain on a triangle with equal edges.
- More than mathematical curiosity: Trojans of Jupiter (N>2000), Mars + Eureka + 3T, Neptun + 5 T




Saturn‘s natural laboratory:
Thetys (1060 km) + Telesto, Calypso (26 km)
Dione (1118 km) + Helene (32 km),
Janus + Epimetheus (horseshoe orbit)

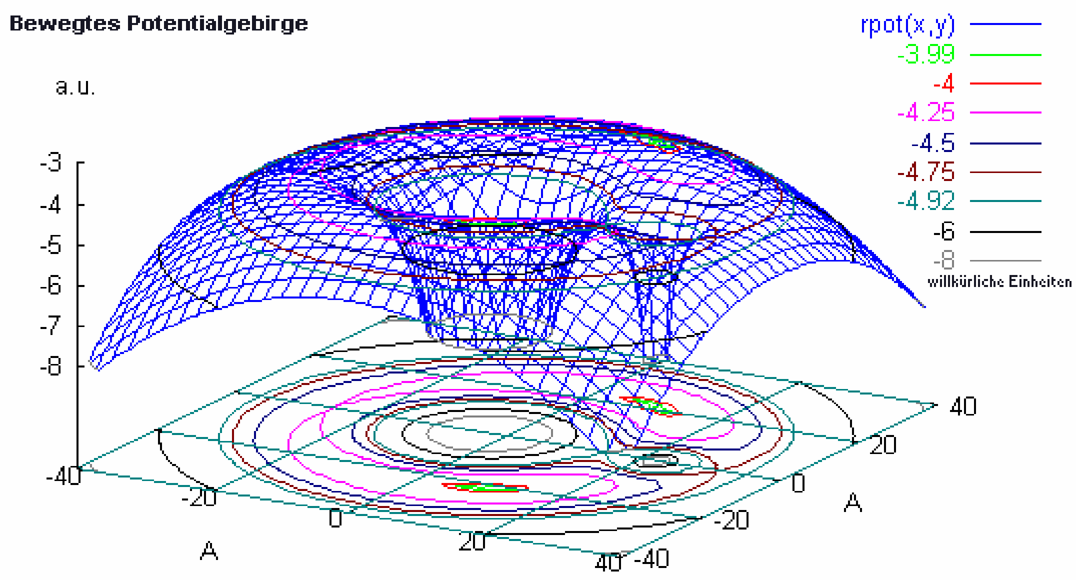


Application for space-probes: (eternal sun or shadow)
SOHO ($L_1$), WMAP ($L_2$), HERSCHEL+PLANCK ($L_2$), JWST ($L_2$), DARWIN ($L_2$ ?)
Trojan Exoplanets

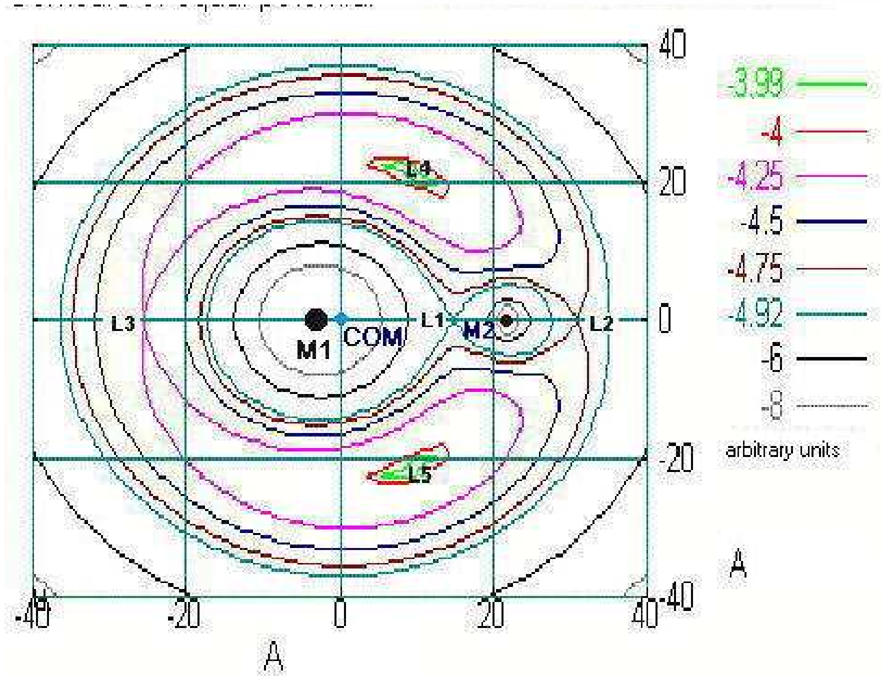
# Lagrange Points

- **Effective Potential:**

$$V(\vec{r}) = -\frac{GM_1}{|\vec{r}-\vec{R}_1|} - \frac{GM_2}{|\vec{r}-\vec{R}_2|} - \frac{\omega^2 r^2}{2}$$




- $r$: Distance to COM,
  $\omega$: angular speed,
  $G$: gravitational constant
- Stable positions: $L_4$, $L_5$
  => natural place for Trojans
- Instable positions: $L_1$, $L_2$, $L_3$
  => for space-probes with correction motors
- Richard Greenberg & Donald Davis: 'Stability at potential maxima: The L4 and L5 points of the restricted three-body problem' in American Journal of Physics, 46, Vol 10, Oct. 1978.



# Orbits in rotating frame


- Yellow: Sun, white point: Jupiter. magenta: disburbed Object at L4. blue, white, red: Objects placed at different distances to Lagrangian Point L5

# 5. Simulation

http://www.physik.uni-kl.de/urbassek/research/anders/Gravity/Gravitation.html

# 6. Literature


- Richard Greenberg & Donald Davis: 'Stability at potential maxima: The L4 and L5 points of the restricted three-body problem' in American Journal of Physics, 46, Vol 10, Oct. 1978. p 1068 ff
- CD with Kepler-ellipse movie in Kaufmann and Freedmann, Universe,1999
- Keller, Kompendium der Astromomie, Stuttgart 2008
- Press et al., Numerical Recipes in C, Cambridge 1992, Reprint 1999
- Sterne und Weltraum 1/2008 (Planck + Herschel mission)
- Lecture Prof. Urbassek Solar System WS 06/07
- Trojan Exoplanets: http://www.trojanplanets.appstate.edu